Max-Planck-Institut für Mathematik Bonn

On a problem involving strongly orthogonal roots

by

Qëndrim R. Gashi

Max-Planck-Institut für Mathematik
Preprint Series 2016 (17)

On a problem involving strongly orthogonal roots

Qëndrim R. Gashi

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

University of Prishtina
Nëna Tereze p.n.
10000 Pristina
Kosovo

MPIM 16-17

ON A PROBLEM INVOLVING STRONGLY ORTHOGONAL ROOTS

QËNDRIM R. GASHI

Abstract. Fixing a natural number k and a root system R, we examine the maximal number of sets of k mutually strongly orthogonal roots so that any two such distinct sets have the property that the difference between their respective sum of all elements can itself be written as a sum of k roots that are mutually strongly orthogonal. The problem that we address is derived from the open question of (non-)existence of finite projective planes, which can be interpreted as belonging to the root system of type A. We formulate the general problem for all root systems and provide results in certain cases.

Contents

Introduction

The notion of strongly orthogonal roots first appeared in the classical works of Harish-Chandra [HC56], in the study of holomorphic discrete series representations, and of Kostant [Kos55], in the study of conjugacy classes of real Cartan subalgebras (see also Sugiura [Sug59]). It has since appeared in many articles from different mathematical fields (cf. [Moo64], [Sch75], [Oh98], [Her01], [Pas01], [Mul02], [EHW04], [MR06], [Kal11], [Mil11], [BGP11], [Kos13]). Some of the properties of strongly orthogonal roots have been studied in detail by Agaoka and Kaneda [AK02].

Our aim in this paper is to study a problem involving strongly orthogonal roots. Fixing a natural number k, we will examine the differences between sums of k roots that are mutually strongly orthogonal. We would like to understand, given a root system R, what is the maximal number $\mu_k(R)$ of sets of k mutually strongly orthogonal roots so that any two such distinct sets have the property that the difference between their respective sum of all elements can itself be written as a sum of k roots that are mutually strongly orthogonal. We state precisely this problem in Subsection 1.3 below.

The problem that we study is derived from the open problem of (non-)existence of finite projective planes. We will see in Section 2 how exactly to pass from incidence matrices of finite projective planes to a problem involving strongly orthogonal roots in the root system of type A. Our formulation of results is a generalization to all root systems.

Our current results do not immediately shed light on the veracity of the old conjecture that finite projective planes exist only for orders of prime power. However, if we were to know that, for a given k, we have $\mu_k(R) < k(k+1)$ for $R = A_{k(k+1)}$, then we would be able to conclude that a finite projective plane of order k does not exist. In any case, since we do not have a complete understanding of the numbers $\mu_k(R)$ for $R = A_\ell$, but only for ℓ large enough and depending of k, we have to be satisfied at the moment only with a conjectural relation between our work and the old conjecture.

This paper is organized as follows. We state our results, after including preliminary notions, in the first section. Next, we explain how to relates those results to the existence of finite projective planes. The last section is devoted to the proof of the main theorem.

Acknowledgments. The author is grateful to the Max Planck Institute for Mathematics in Bonn for its hospitality and the excellent working conditions, where parts of this paper were written, while on a sabbatical leave from University of Prishtina. He is also indebted to Bujar Shita for stimulating conversations and Artan Berisha for help with a computer program. Finally, the author thanks Iain Gordon and the London Mathematical Society for a grant that enabled his stay at the University of Edinburgh, where this project was first started.

1. Statement of results

1.1. **Root systems.** We recall the definition of a root system (for details, cf. [Bou02]). Let V be a vector space over $\mathbb{R}$ and let $(\,,)$ be an inner product on V. A finite subset R of non-zero elements of V is called a root system if it satisfies the following:

(R1) R spans V;
(R2) For any $\alpha, \beta \in R$, we have $m_{\alpha,\beta} = 2\frac{(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$;
(R3) For any $\alpha, \beta \in R$, we have $\beta - m_{\alpha,\beta}\alpha \in R$.

Elements of R are called roots. If for any $\alpha \in R$ and $k \in \mathbb{Z}$ we have $k\alpha \in R$ only for $k = \pm 1$, then we say that R is reduced. The root system R is called irreducible if it is not the direct sum of two root systems. The rank of R is the dimension of the subspace $\mathbb{R}(R)$ that is spanned by all roots.

Irreducible, reduced root systems are of the following types:

$$A_\ell, B_\ell, C_\ell, D_\ell, E_6, E_7, E_8, F_4 \text{ or } G_2,$$

where the index denotes the corresponding rank.

1.2. **Strongly orthogonal roots.** In a given a root system R, two distinct roots α and β are said to be strongly orthogonal if neither $\alpha + \beta$ nor $\alpha - \beta$ is a root. One can check immediately that strongly orthogonal roots are also orthogonal with respect to the initial inner product, but the converse does not hold in general (although in type A the two notions coincide).

1.3. **Statement of the problem.** Let R be a root system. A subset of R consisting of mutually-strongly orthogonal roots is called a strongly orthogonal subset (SOS). Denote the set of all SOS's in R that have exactly k-elements by $SOS_k(R)$. If $\Gamma \in SOS_k(R)$, we denote by $|\Gamma|$ the sum of the elements of Γ.

We would like to understand the following general problem: Find the maximal number $\mu = \mu_k(R)$ such that there exist $\Gamma_i \in SOS_k(R)$ $(i = 1, \ldots, \mu)$ with the property that for any two distinct i, j there exists $\Gamma_{i,j} \in SOS_k(R)$ such that

$$|\Gamma_i| - |\Gamma_j| = |\Gamma_{i,j}|. \tag{S}$$

Note that for $k = 1$ we are posing the question of finding the cardinality of the longest sequence (γ_i) of roots of R such that $\gamma_i - \gamma_j$ is also a root for all $i \neq j$.

1.4. **Results.** Let us first remark that we already know when $\mu_k(R)=0$ from the fact that maximal strongly orthogonal subsets have already been studied and it was established when $SOS_k(R)=\emptyset$. The point is to calculate $\mu_k(R)$ in other cases. Let us recall the known results about $\mu_k(R)$:

Proposition 1.1. (*cf.* [AK02], [BGP11]) *The number $\mu_k(R)$ is zero in the following and only following cases:*

(i) *For $k > \left\lfloor \frac{\ell+1}{2} \right\rfloor$ when $R = A_\ell$;*
(ii) *For $k > \ell$ when $R = B_\ell$;*
(iii) *For $k > \ell$ when $R = C_\ell$;*
(iv) *For $k > 2\left\lfloor \frac{\ell}{2} \right\rfloor$ when $R = D_\ell$;*
(v) *For $k > 3$ when $R = F_4$;*
(vi) *For $k > 4$ when $R = E_6$;*
(vii) *For $k > 7$ when $R = E_7$;*
(viii) *For $k > 8$ when $R = E_8$;*
(ix) *For $k > 2$ when $R = G_2$.*

In the case of root systems of type A, we understand the numbers $\mu_k(A_\ell)$ for ℓ sufficiently large.

Theorem 1.2. *Fix $k \in \mathbb{N}$. There exists $N = N(k) \in \mathbb{N}$ such that*

$$\mu_k(A_\ell) = \left\lfloor \frac{\ell-(k-1)}{k} \right\rfloor$$

holds for all $\ell \geq N$.

For $k=1$ or 2, we have a complete answer

Theorem 1.3. *We have that $\mu_1(A_l) = l$. The sequence $\mu_2(A_l)$, with $l \in \mathbb{N}$, is equal to $\lfloor (l-1)/2 \rfloor$ for $l \geq 13$, and for $l < 13$ its terms are $0,0,1,1,3,6,6,6,6,6,6,6$.*

A remark is in order on theorems 1.2 and 1.3. It is not unusual for some problems in mathematics to encounter "obstruction" in low dimensions. The question we have posed seems to fit into that category. So, for a fixed k, we have an "erratic" behavior of $\mu_k(A_\ell)$ for small values of ℓ and eventually that sequence stabilizes for large enough ℓ.

The proof of Theorem 1.3 is a matter of tedious calculations. We also skip the proofs for the following results for other root systems.

Theorem 1.4. (B1) *For $\ell \gg 0$, $\mu_1(B_\ell) = \ell$.*

(B2) *For $\ell \gg 0$,*

$$\mu_2(B_\ell) = \begin{cases} \ell-2 & \text{if } \ell \text{ is even} \\ \ell-1 & \text{if } \ell \text{ is odd.} \end{cases}$$

(C1) *For $\ell \gg 0$, $\mu_1(C_\ell) = 2(\ell-1)$.*
(C2) *For $\ell \gg 0$, $\mu_2(C_\ell) = \ell-1$.*
(D1) *$\mu_1(D_\ell) = \ell-1, \forall \ell \geq 3$.*
(D2) *For $\ell \gg 0$,*

$$\mu_2(D_\ell) = \begin{cases} \ell-1 & \text{if } \ell \text{ is even} \\ \ell-2 & \text{if } \ell \text{ is odd.} \end{cases}$$

(D3) *For $\ell \gg 0$, we have*

$$\mu_3(D_\ell) = \begin{cases} 4\left\lfloor \frac{\ell-4}{4} \right\rfloor + 2 & \text{if } \ell \equiv 3 (mod 4), \\ 4\left\lfloor \frac{\ell-4}{4} \right\rfloor + 1 & \text{otherwise.} \end{cases}$$

(D4) *For* $\ell \gg 0$, $\mu_4(D_\ell) = \ell - 1$.

(E) $\mu_4(E_6) = 2$, $\mu_k(E_6) = 0$ *for* $k \geq 5$.

(F) $\mu_1(F_4) = 4$, $\mu_2(F_4) = 3$, $\mu_3(F_4) = 3$, $\mu_4(F_4) = 3$, *and* $\mu_k(F_4) = 0$ *for* $k \geq 5$.

(G) $\mu_1(G_2) = 3$, $\mu_2(G_2) = 2$, *and* $\mu_k(G_2) = 0$ *for* $k \geq 3$.

2. Relation with the existence of finite projective planes

2.1. **Finite projective planes.** Let us recall that a finite projective plane of order n ($n \in \mathbb{N}$) is a collection of lines and points such that:

- every line contains $n+1$ points,
- every point is on $n+1$ lines,
- any two distinct lines intersect at exactly one point, and
- any two distinct points lie on exactly one line.

From these conditions it follows that if a plane of order n exists, then there must be exactly n^2+n+1 points and n^2+n+1 lines. The existence of finite projective planes of order n can be reduced to the existence of the corresponding incidence matrix $X = (x_{ij})$, where

(i) X is a square matrix of order n^2+n+1,

(ii) For each i, j we have $x_{i,j} \in \{0,1\}$,

(iii) The sum of all entries in any row as well as in any column is $n+1$,

(iv) The inner product of any two distinct rows as well as of any two distinct columns is 1.

A well-known conjecture states that finite projective planes exist only for prime power order. The conjecture remains open, although there are some quite general partial results, like the Bruck-Ryser Theorem ([BR49]) which states that if a finite projective plane of order n exists and n is congruent to 1 or 2 ($mod\, 4$), then n can be written as a sum of two squares. Many years ago C. Lam proved ([Lam91]) the non-existence of projective planes of order 10 using a computer, but even the case $n = 12$ remains elusive thus far.

2.2. **Partial reformulation using root systems.** Let us start with an illustrative example. The incidence matrix, up to permutation, of the projective plane of order 2 is

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

If we fix the first row and take its difference with each of the other rows, we get the following matrix

$$\begin{pmatrix} 0 & 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 & -1 \\ 1 & 0 & 1 & -1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 & -1 \\ 1 & 1 & 0 & -1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 & -1 & -1 & 0 \end{pmatrix}$$

Considering rows as vectors inside $\mathbb{R}^7$, we get the following six vectors in the lattice of weights of A_6, which in the Bourbaki notation correspond to the following sums of two strongly orthogonal roots: $(\alpha_2+\alpha_3+\alpha_4)+\alpha_2$, $(\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6)+(\alpha_3+\alpha_4+\alpha_5)$, $(\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5)+\alpha_3$, $(\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6)+(\alpha_3+\alpha_4)$, $(\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6)+(\alpha_2+\alpha_3)$, and

$(\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5)+(\alpha_2+\alpha_3+\alpha_4)$. In other words, we get six $SOS_2(A_6)$'s. One can check easily that they satisfy the property from Subsection 1.3.

In general, if a finite projective plane of order n exists, then we can consider its incidence matrix, take the differences between the first (or any fixed row) and all other rows. We get $n(n+1)$ elements of $SOS_n(A_{n(n+1)})$ which satisfy the property from Subsection 1.3. Thus, in this case, $\mu_n(A_{n(n+1)}) \geq n(n+1)$. And we conclude that if $\mu_n(A_{n(n+1)}) < n(n+1)$, then there is no finite projective plane of order n.

We can formulate the following

Question 2.1. *For n not equal to a prime-power, is it true that $\mu_n(A_{n(n+1)}) < n(n+1)$?*

As mentioned already, in this paper we attempt to better understand the sequences $\mu_k(R)$ not just for the root systems of type A, but in general. Much remains to be done to understand these sequences for any root system. One could attempt to tackle this problem not just from a combinatorial perspective, but also from a geometric perspective, making use of the fact (cf. [Sch75, pg. 55]) that strong-orthogonality is a condition implying the commutativity of Cayley transforms between Cartan subalgebras. But, that is beyond the scope of the current paper.

3. Proof of Theorem 1.2

Fix $k \in \mathbb{N}$. We aim to prove that for sufficiently large $\ell \in \mathbb{N}$ we have $\mu_k(A_\ell) = \left\lfloor \frac{\ell-(k-1)}{k} \right\rfloor$. Note that (cf. Proposition 1.1) if $k > \lfloor \frac{\ell+1}{2} \rfloor$, then $\mu_k(A_\ell) = 0$. We will therefore assume throughout this subsection that $k \leq \lfloor \frac{\ell+1}{2} \rfloor$.

Let us first observe that roots of A_ℓ are $\varepsilon_i - \varepsilon_j$, with $1 \leq i \neq j \leq \ell+1$, and where (ε_i) is the canonical basis of $\mathbb{R}^\ell$. A natural basis of roots of A_ℓ is given by $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $(i = 1, 2, \ldots, \ell)$.

Next, observe that two roots $\varepsilon_i - \varepsilon_j$ and $\varepsilon_p - \varepsilon_q$ are strongly orthogonal if and only if $i, j \notin \{p, q\}$ (and, of course, $i \neq j$ and $p \neq q$ or else $\varepsilon_i - \varepsilon_j$ and $\varepsilon_p - \varepsilon_q$ would not be roots in the first place).

Fix an element $\Gamma_1 \in SOS_k(A_\ell)$. Then the sum $|\Gamma_1|$ of elements Γ_1 is a vector from $\mathbb{Z}^{\ell+1}$ with exactly $2k$ coordinates that are nonzero, where k of them are equal to 1 and the other k equal to -1.

After permutation, we can assume without loss of generality that the elements of Γ_1 are the following $\alpha_1+\alpha_2+\ldots+\alpha_{2k-1}, \alpha_2+\alpha_3+\ldots+\alpha_{2k-2}, \alpha_3+\alpha_4+\ldots+\alpha_{2k-3}, \cdots, \alpha_{k-1}+\alpha_k+\alpha_{k+1}$, and α_k. This implies that the sum $|\Gamma_1|$ is equal to $\sum_{i=1}^k \varepsilon_i - \sum_{i=k+1}^{2k} \varepsilon_i$.

Suppose that there exists $\Gamma_2 \in SOS_k(A_\ell)$ such that property (S) from subsection 1.3 holds for Γ_1 and Γ_2. Denote by $i_1 < i_2 < \ldots < i_{2k-1} < i_{2k}$ the ordered non-zero coordinates of $|\Gamma_2| \in \mathbb{Z}^{\ell+1}$. Again, k of them are equal to 1 and the other k equal to -1.

Property (S) for Γ_1 and Γ_2 implies that the two vectors $|\Gamma_1|$ and $|\Gamma_2|$ overlap on exactly k non-zero coordinates (and are both zero on the same $\ell+1-3k$ zero coordinates). Therefore, $i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, 2k-1, 2k\}$ and $i_{k+1} > 2k$. Without loss of generality, by permuting coordinates if necessary, we can assume that $i_{k+1} = 2k+1, i_{k+2} = 2k+2, \ldots, i_{2k} = 3k$.

Suppose now that there exists, in addition, $\Gamma_3 \in SOS_k(A_\ell)$ such that property (S) from subsection 1.3 holds for $\Gamma_1, \Gamma_2, \Gamma_2$. Denote by $j_1 < j_2 < \ldots < j_{2k-1} < j_{2k}$ the ordered non-zero coordinates of $|\Gamma_3| \in \mathbb{Z}^{\ell+1}$. Then, again we have $j_1, j_2, \ldots, j_k \in \{1, 2, \ldots, 2k-1, 2k\}$ and $j_{k+1} > 2k$.

There are obviously two possibilities:

(i) All the three vectors $|\Gamma_1|, |\Gamma_2|, |\Gamma_3|$ overlap on the same k coordinates.
(ii) Not all the three vectors $|\Gamma_1|, |\Gamma_2|, |\Gamma_3|$ overlap on the same k coordinates.

We will now prove that in case (ii), for ℓ large enough (recall that we have fixed k), the maximal number of elements of $SOS_k(A_\ell)$ satisfying (S) has an absolute upper bound that does not depend on ℓ.

Suppose that (ii) holds. Since not all the three vectors $|\Gamma_1|, |\Gamma_2|, |\Gamma_3|$ overlap on the same k coordinates, but each pair of such vectors overlaps on exactly k coordinates, we infer that at least $k+1$ non-zero coordinates of Γ_3 must lies on the first $3k$ coordinates. The same can be inferred about any element of $SOS_k(A_\ell)$ that we would consider to construct a sequence of Γ_i's as in subsection 1.3.

But, in order to ensure that (S) is satisfied, no two elements Γ_i should overlap on more than k non-zero coordinates. This yields an upper bound for such a sequence $\{\Gamma_i\}$ since we have to choose $k+1$ entries from the first $3k$ coordinate and there should be no overlap on more than k coordinates for any two $|\Gamma_i|'s$. The crude estimate is that in this case the sequence $\{\Gamma_i\}$ contains at most $\binom{3k}{k+1}+2$ elements. In reality this upper bound is much smaller, but we do not need that fact in the proof of Theorem 1.2.

It remains that, in order to construct $\mu_k(A_\ell)$ for sufficiently large ℓ, we only need to deal with the situations of case (i) above. In that case, we can immediately check that $\mu_k(A_\ell) = \left\lfloor \frac{\ell-(k-1)}{k} \right\rfloor$, which we can get explicitly from the sequence of Γ_i's with Γ_p consisting of the following k roots:

$$\begin{aligned}
&\alpha_1 + \alpha_2 + \ldots + \alpha_{pk},\\
&\alpha_2 + \alpha_3 + \ldots + \alpha_{pk-1},\\
&\ldots\\
&\alpha_{k-1} + \alpha_k + \ldots + \alpha_{(p-1)k+1} + \alpha_{(p-1)k+2},\\
&\alpha_k + \alpha_{k+1} + \ldots + \alpha_{(p-1)k+1}.
\end{aligned}$$

For illustration, we have written below a maximal sequence in a matrix format, with rows representing $|\Gamma_i|$'s. Here we have taken $k=3$ and ℓ is sufficiently large (congruent to 1 mod 3, but the last restriction in not essential):

$$\begin{pmatrix}
1 & 1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0\\
1 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0\\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & ... & 0 & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots\\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ... & -1 & -1 & -1 & 0
\end{pmatrix}.$$

This completes the proof of Theorem 1.2.

References

[AK02] Y. Agaoka and E. Kaneda, *Strongly orthogonal subsets in root systems*, Hokkaido Math. J. **31** (2002), 107–136.

[BGP11] Lucio Bedullia, Anna Gorib, and Fabio Podestà, *Homogeneous hyper-complex structures and the Joyces construction*, Differential Geom. Appl. **29** (2011), no. 4, 547–554.

[Bou02] N. Bourbaki, *Elements of Mathematics, Lie groups and Lie algebras, Chapters 4-6*, Springer-Verlag, Berlin Heidelberg, 2002.

[BR49] R. H. Bruck and H. J. Ryser, *The nonexistence of certain finite projective planes*, Canad. J. Math. **1** (1949), 88–93.

[EHW04] T. J. Enright, M. Hunziker, and N. R. Wallach, *A Pieri rule for Hermitian symmetric pairs I*, Pacific J. Math. **214** (2004), no. 1, 23–30.

[HC56] Harish-Chandra, *Representations of semi-simple Lie groups VI*, American Journal of Mathematics **78** (1956), 564–628.

[Her01] R. A. Herb, *Discrete series characters as lifts from two-structure groups*, Trans. Amer. Math. Soc. **353** (2001), 2557–2599.

[Kal11] T. Kaletha, *Decomposition of splitting invariants in split real groups*, Canad. J. Math. **63** (2011), no. 5, 1083–1106.

[Kos55] B. Kostant, *On the conjugacy of real Cartan subalgebras. I*, PNAS **41** (1955), no. 11, 967–970.

[Kos13] ———, *Center U(n) , Cascade of orthogonal roots, and a construction of Lipsman-Wolf*, Lie Groups: Structure, Actions, and Representations (A. Huckleberry et al., ed.), Progress in Mathematics, no. 306, Springer New York, 2013, pp. 163–173.

[Lam91] C. Lam, *The search for a finite projective plane of order 10*, The American Mathematical Monthly **98** (1991), no. 4, 305–318.

[Mil11] T. Milev, *Root Fernando-Kac subalgebras of finite type*, J. Algebra **336** (2011), no. 1, 257–278.

[Moo64] C. C. Moore, *Compactifications of symmetric spaces II: The Cartan domains*, American Journal of Mathematics **86** (1964), no. 2, 358–378.

[MR06] H. B. Messaoud and G. Rousseau, *Sous-algèbres de Cartan des algèbres de Kac-Moody réelles presque déployées*, J. Math. Soc. Japan **58** (2006), no. 4, 1009–1030.

[Mul02] I. Muller, *Racines orthogonales, formes quadratiques et orbites d'algèbres de Lie simples graduées*, J. Algebra **250** (2002), no. 1, 288–334.

[Oh98] H. Oh, *Tempered subgroups and representations with minimal decay of matrix coefficients*, Bull. Soc. Math. France **126** (1998), 355–380.

[Pas01] D. Pascovici, *The Dynkin diagram R-group*, Represent. Theory **5** (2001), 1–16.

[Sch75] W. Schmid, *On the characters of the discrete series, The Hermitian symmetric case*, lnventiones math. **30** (1975), 47–144.

[Sug59] M. Sugiura, *Conjugate classes of Cartan subalgebras in real semisimple Lie algebras*, J. Math. Soc. Japan **11** (1959), no. 4, 374–434.

Univeristy of Prishtina, Nëna Tereze p.n., 10000 Pristina, Kosovo
Current address: Max Planck Institute for Mathematics, Vivatsgasse 7, 53111 Bonn, Germany
E-mail address: `qendrim@mpim-bonn.mpg.de, qendrim@gmail.com`