LOOPING OF THE NUMBERS GAME AND THE ALCOVED HYPERCUBE

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Abstract. We study the so-called *looping* case of Mozes's game of numbers, which concerns the (finite) orbits in the reflection representation of affine Weyl groups situated on the boundary of the Tits cone. We give a simple proof that all configurations in the orbit are obtainable from each other by playing the numbers game, and give a strategy for going from one configuration to another. This strategy gives rise to a partition of the finite Weyl group into finitely many graded posets, one for each extending vertex of the associated extended Dynkin diagram. These posets are selfdual and mutually isomorphic, and their Hasse diagrams are dual to the triangulation of the unit hypercube by reflecting hyperplanes. Unlike the weak and Bruhat orders, the top degree is cubic in the number of vertices of the graph. We explicitly compute the rank generating function of the poset.

Contents

1. Introduction

1.1. **The numbers game.** Mozes's game of numbers [Moz90], which originated from (and generalizes) a 1986 IMO problem, has been widely studied (cf. [Pro84, Pro99, DE08, Eri92, Eri93, Eri94a, Eri94b, Eri95, Eri96, Wil03a, Wil03b]), and yields useful algorithms for computing with the root systems and reflection representations of Coxeter groups (see [BB05, §4.3] for a brief summary).

We briefly recall the numbers game. Consider a Coxeter group W with generators $s_i, i \in I$, and relations $(s_i s_j)^{n_{ij}} = 1$, for $n_{ij} = n_{ji} \in \mathbb{Z}_{\geq 1} \sqcup \{\infty\}$ ($n_{ii} = 1$ for all i, and $n_{ij} \geq 2$ for $i \neq j$). We associate to this an unoriented graph Γ with no loops and no multiple edges, with vertex set I, such that two vertices i, j are adjacent if $n_{ij} \geq 3$. Consider also a choice of Cartan matrix $C = (c_{ij})$ such that $c_{ii} = 2$ for all i, $c_{ij} = 0$ whenever i and j are not adjacent, and in the case i, j are adjacent, $c_{ij}, c_{ji} < 0$ and either $c_{ij}c_{ji} = 4\cos^2(\frac{\pi}{n_{ij}})$ (when n_{ij} is finite), or $c_{ij}c_{ji} \geq 4$ (when $n_{ij} = \infty$).

The *configurations* of the game consist of vectors $\mathbb{R}^I$, considered as labelings of the vertices of the graph Γ. The moves of the game are as follows: for any vector $v = (v_i)_{i \in I} \in \mathbb{R}^I$ and any vertex $i \in I$ such that $v_i < 0$, one may perform the following move, called *firing the vertex* i: v is replaced by the new configuration $s_i(v)$, defined by

$$(s_i(v))_j = \begin{cases} -v_i, & \text{if } j = i, \\ v_j - c_{ij}v_i, & \text{if } j \text{ is adjacent to } i, \\ v_j, & \text{otherwise.} \end{cases} \tag{1.1}$$

The entries v_i of the vector v are called *amplitudes*. The game terminates if all the amplitudes are nonnegative. Let us emphasize that *only vertices with negative amplitudes may be fired.*

The following summarizes some basic known results. Here $\cdot$ denotes the usual dot product in $\mathbb{R}^I$. In the case where the Coxeter group is an affine Weyl group and $C = C_0$ is the standard integral Cartan matrix, the vector $\delta \in \mathbb{Z}^I_{\geq 1}$ is a generator of the kernel of the Cartan form, expressed in the basis of simple roots; the precise definition is recalled in Section 2.

Theorem 1.2. (i) [Moz90, Eri96] *If the numbers game terminates, then it must terminate in the same number of moves and at the same configuration regardless of how it is played.*
(ii) *In the finite Coxeter group case, the numbers game must terminate.*
(iii) [Eri94a] *In the affine case with $C = DC_0D^{-1}$ for D a diagonal matrix with positive diagonal entries,*[1] *the numbers game terminates if and only if $\delta \cdot (Dv) > 0$.*
(iv) [Eri94a] *Whenever the numbers game does not terminate, it reaches infinitely many distinct configurations, except for the affine case with $C = DC_0D^{-1}$ and $\delta \cdot (Dv) = 0$ (but $v \neq 0$), in which case only finitely many configurations are reached (i.e., the game "loops").*

In particular, we will be concerned here with the *looping* case. From now on, we assume therefore that W is an affine Weyl group. Let Γ be the underlying undirected graph of the Dynkin diagram of W, obtained by forgetting edge directions and multiplicities. Let $\Gamma_0 \subset \Gamma$ be a subgraph obtained by removing a vertex, which is the underlying undirected graph of a finite Dynkin diagram from which the Dynkin diagram of W is obtained by adjoining an extending vertex. Let $W_0 \subseteq W$ be the finite Weyl group associated to Γ_0, generated by s_i for i ranging over the vertices of Γ_0.

We explain briefly how our results relate to standard constructions in the theory of Coxeter groups. The configuration space $\mathbb{R}^I$ is the reflection representation of the Coxeter group W. The subset of $\mathbb{R}^I$ where the numbers game terminates is called the Tits cone. The Tits cone is naturally divided into simplicial cones, with the maximal cones labeled by the elements of W (see [BB05, Sections 4.3 and 4.9]). Starting with a point in the interior of one of these maximal cones, the cones we travel through form a descending chain in the *weak order* (whose definition we recall in §6); the restriction of firing only negative amplitudes means that we only move downward. So results (i) and (ii), in part, say that weak order is graded and has a unique minimal element.

Our results study the affine case when the initial vector is not within the Tits cone, but rather at its boundary. In the affine case, the boundary of the Tits cone is a hyperplane, divided into finitely many simplicial cones. These cones are indexed by the elements of W_0. However, our problems do not reduce to the numbers game on W_0, but instead reveal several new and interesting combinatorial structures. That is, some combinatorics of W appear inside W_0, in a sense we will explain.

[1]In all affine types other than $\widetilde{A_n}$, since the graph Γ is acyclic, C is automatically of this form.

1.2. **Motivation and results.** The original motivation of this paper was the following question: in the looping case, can one always return to the initial configuration? This was asserted to be true in [Eri94a],[2] but a proof was not provided. Our first goal is to provide a simple proof in the affirmative. In fact, we prove more: we give in §3 a *strategy* for going from any configuration v to any element of its Weyl orbit in a number of moves cubic in the number of vertices, and our strategy is optimal in certain cases.

Following the strategy leads to a graded selfdual poset of obtainable configurations, which we define in §4. To define this, we fix a finite Dynkin subdiagram whose associated extended Dynkin diagram is the original graph. Then, the poset begins with a configuration whose restriction to a fixed finite Dynkin subdiagram is dominant, and ends with one whose restriction is antidominant. Since the choices of finite Dynkin subdiagram correspond to the choices of extending vertices, this decomposes the finite Weyl group associated to our graph canonically into disjoint copies of the aforementioned poset, one for each extending vertex. As we will see, this poset does not closely resemble a poset associated to the finite Weyl group: it has *cubic* degree in the number of vertices, unlike the weak or Bruhat order posets for the finite Weyl group, which have quadratic degree. In the remainder of the paper, we study this poset.

In §5 we show that following our strategy on a looping configuration is equivalent to arbitrarily playing the numbers game on a modified configuration for which the numbers game terminates. The main result here, Proposition 5.1, is a key technical ingredient in the remaining results.

In §6 we identify the aforementioned poset with an interval under the (left) weak order in the affine Weyl group. Moreover, we show in §7 that the Hasse diagram of the poset coincides with the dual of the triangulation of the unit hypercube in the reflection representation of the affine Weyl group. This triangulation has been studied in many places, notably recently in [LP07] in type A.

Finally, in §8, we explicitly compute the rank generating function[3] of the poset: this is a polynomial whose degree bounds the number of moves required by the main part of our aforementioned strategy for going from one looping configuration to another. In particular, this proves that the top degree is cubic in the number of vertices. The coefficients in each degree say how many different configurations can be obtained by following our strategy, starting with the initial configuration in lowest degree of the poset, for a number of moves equal to the degree. Going from the initial configuration to the final one gives rise to a canonical involution of the extended Dynkin diagram which we also compute. In §8.3, we give a combinatorial interpretation and proof of this formula in the type A cases (our proof in general type relies on the formulas for the rank generating functions of W and W_0).

Evaluating the rank generating function (which is a polynomial) at 1 in two ways yields a curious identity (which was unknown to us): Let m_i be the Coxeter exponents of W_0. Let I be the vertex set of Γ. Then,

$$\prod_i (m_i(m_i+1)) = \#(\text{extending vertices of } \Gamma) \cdot \prod_{i \in I} L(t_i), \tag{1.3}$$

where the elements $t_i \in W$ are those that take the dominant chamber of $\mathcal{H} := \{v \in \mathbb{R}^I \mid \delta \cdot v = 1\}$ (i.e., the locus where all coordinates are nonnegative) to its translate by $\omega^i - \delta_i \omega^j$, where ω^i is the i-th fundamental coweight (which is the i-th basis vector of our configuration space $\mathbb{R}^I$), and $L(t_i)$ is the length of t_i, i.e., the minimum number of simple reflections whose product is t_i.

[2] Also, there it was asserted that there is a way to play the numbers game that passes through all configurations in the Weyl orbit of the vector v exactly once (i.e., that a Hamiltonian cycle exists in the directed graph whose vertices are this Weyl orbit and directed edges are moves of the numbers game). We do not have either a proof or counterexample to this assertion.

[3] The rank generating function is also known as the Hilbert series (which is a polynomial here since the poset is finite).

1.3. **Acknowledgements.** We thank T. Lam for essential discussions about the rank generating functions of our posets. While the research for this project was conducted, the first author was an EPDI fellow, the second author an AIM fellow, and the third author a Clay research fellow. The first two authors were also supported by Clay Liftoff fellowships, and the second author by the University of Chicago's VIGRE grant. We thank the University of Chicago, MIT, and the Isaac Newton Institute for Mathematical Sciences for hospitality.

2. Preliminaries on affine Weyl groups

2.1. **Recollections.** For a reference on the material in this subsection, see, e.g., [Bou02, §VI.4]. By a *Dynkin graph*, we mean the underlying undirected graph of a Dynkin diagram, without multiple edges. That is, in a Dynkin graph, any two adjacent vertices are connected by a single undirected edge. Let us make the same definition for extended Dynkin graphs.

In this note, we will be concerned with the numbers game when Γ is an extended Dynkin graph with vertex set I (for $|I| \geq 2$), and C is the standard integral Cartan matrix associated to the extended Dynkin diagram (see Remark 2.2 below). Then, the Coxeter group W is the affine Weyl group associated to the extended Dynkin diagram.

We will make use of the root systems associated to Dynkin and extended Dynkin graphs. Let Δ and Δ_+ be the sets of roots and positive roots, respectively. We will view Δ and $\Delta_+ \subset \mathbb{Z}^I$ in the basis of simple roots. Then, the dot product between roots and configuration vectors (viewed as coweights) is the canonical pairing.

To be precise, by roots we mean what are sometimes called real roots, i.e., the images of the simple roots under the Coxeter group action dual to the action on the space of configurations: $s_i\alpha = \alpha - \langle \alpha, \alpha^i \rangle \alpha^i$, where $\alpha^i \in \Delta_+ \subset \mathbb{Z}^I$ is the i-th simple root and $\langle\, , \rangle$ is the Cartan form, $\langle \alpha^i, \alpha^j \rangle = c_{ij}$.

Recall that, for a (finite) Dynkin diagram, one can form the associated extended Dynkin diagram by adjoining a new vertex corresponding to the negative of the maximal root β: in other words, β is the unique positive root such that $\langle \beta, \alpha^j \rangle \geq 0$ for all simple roots α^j. Conversely, let us call a vertex of an extended Dynkin diagram *extending* if its removal results in a Dynkin diagram whose extended Dynkin diagram is the original diagram. For example, all vertices of a type $\widetilde{A}_n$ extended Dynkin diagram are extending, whereas the four external vertices of a type $\widetilde{D}_n$ extended Dynkin diagram are extending.

Passing to the underlying extended Dynkin graph Γ, call a vertex extending if it was an extending vertex of the extended Dynkin diagram.

Finally, we will make use of the element $\delta \in \mathbb{Z}_+^I$, uniquely given so that $\delta_i = 1$ at all extending vertices of the graph, and $\langle \delta, \alpha \rangle = 0$ for all $\alpha \in \Delta$. We write $\delta^\perp \subseteq \mathbb{R}^I$ for the space of configurations orthogonal to δ; this is the boundary of the Tits cone.

We emphasize that Δ is the set of roots for the affine Weyl group. On the occasion that we need to refer to the root system of the associated finite Coxeter group, we write Δ^0; then Δ_+^0 will denote the subset of positive roots in Δ^0.

2.2. **Connection to the numbers game.** The following proposition explains the connection between roots and the numbers game.

Proposition 2.1. *Start with any configuration v and perform a valid sequence of firing moves. If b is an amplitude (necessarily negative) which is fired at some point then there is a positive root β such that $\beta \cdot v = b$.*

Specifically, if the sequence is $i_1, \ldots, i_m$, then the word $s_{i_m} \cdots s_{i_1}$ is reduced, and the j-th amplitude fired (at vertex i_j) is $(s_{i_1} s_{i_2} \cdots s_{i_{j-1}} \alpha^{i_j}) \cdot v$.

Proof. The final assertion, that the j-th amplitude fired is $(s_{i_1} \cdots s_{i_{j-1}} \alpha^{i_j}) \cdot v$, is obvious, since this equals $\alpha^{i_j} \cdot (s_{i_{j-1}} \cdots s_{i_1}(v))$. Moreover, if $s_{i_1} \cdots s_{i_m}$ is reduced, then so is $s_{i_1} \cdots s_{i_j}$, and hence $s_{i_1} \cdots s_{i_{j-1}} \alpha^{i_j}$ is positive. Thus, it suffices to show that $s_{i_1} \cdots s_{i_m}$ is reduced.

Suppose, for sake of contradiction, that this word is not reduced. Let ℓ be the first index such that $s_{i_1} s_{i_2} \cdots s_{i_\ell}$ is not reduced. Then there is some $k < \ell$ such that $s_{i_1} s_{i_2} \cdots s_{i_{k-1}} \alpha^{i_k} = -s_{i_1} s_{i_2} \cdots s_{i_{\ell-1}} \alpha^{i_\ell}$. But then, the k-th amplitude fired is negative the ℓ-th amplitude fired, which is impossible. Thus we have a contradiction. □

Remark 2.2. Although it is convenient to work with the standard integral $C = C_0$ as above, this is not essential. If $C = DC_0D^{-1}$, for D a diagonal matrix with positive entries, then one has a standard equivalence between the numbers game using C_0 and the numbers game using C, by the map on configurations $v \mapsto D^{-1}v$. In particular, the looping configurations for the numbers game using C are those $v \neq 0$ such that $\delta \cdot (Dv) = 0$. This allows one to transplant the results of this paper to the general looping case.

3. Strong looping of the numbers game

In this section, we prove and generalize the following:

Theorem 3.1. *Whenever the numbers game loops, one can always return to the initial configuration.*

The initial configuration v is looping if and only if $\delta \cdot v = 0$ and $v \neq 0$. The theorem can now be re-expressed as: if $\delta \cdot v = 0$, then for every element w of the Coxeter group of the graph, one can go from v to $w(v)$ by playing the numbers game.

As in the introduction, fix a choice of $\Gamma_0 \subset \Gamma$ obtained by removing an extending vertex of Γ. We emphasize that *from now on, all results stated are valid also if we made any other choice of* Γ_0 *from the beginning.* Following the introduction, let W_0 be the Coxeter group associated to Γ_0 and the corresponding restriction of C, which is a finite Weyl group associated to the affine Weyl group W.

Recall that the inclusion of W_0 into W has a standard section $W \to W_0$ (a group homomorphism), and that the action of W on the hyperplane $\delta^\perp$ factors through W_0; therefore, when $v \in \delta^\perp$ and $w \in W_0$, we will feel free to write $w(v)$. Note that, as a consequence, whenever $v \in \delta^\perp$, the orbits $W(v)$ and $W_0(v)$ are identical.

Because we have chosen Γ_0 in Γ, we have a canonical isomorphism between $\delta^\perp$ and the reflection representation of W_0; namely, take a configuration of amplitudes on Γ, lying in $\delta^\perp$, and restrict them to Γ_0 to obtain a configuration on Γ_0.

The reflecting hyperplanes divide $\delta^\perp$ into finitely many chambers, indexed by W_0. We now explain some preliminary reductions, followed by our primary technical result (Theorem 3.3):

Proposition 3.2. *Let v lie in the interior of a chamber C of $\delta^\perp$. Let v' be a point in the W_0 orbit of v, say $v' = w(v)$, and let C' be the chamber in which v' lies. Suppose that we have a sequence of firings which takes v to v'. Let x be another point in C, possibly on the boundary of C. Then, starting with x, there is a subsequence of the preceding sequence of firings (in the case of v) which is legal (beginning with x, only vertices of negative amplitude are fired), and which takes x to $w(x)$.*

Proof. Let the sequence of firings be $s_{i_1}, s_{i_2}, \ldots, s_{i_\ell}$, with the corresponding sequence of chambers $C_1 = s_{i_1}(C), C_2 = s_{i_2}s_{i_1}(C), \ldots, C_\ell = s_{i_\ell} \cdots s_{i_1}(C)$. Define $x_k = s_{i_k} s_{i_{j-1}} \cdots s_{i_1}(x)$. Note that x_k is in C_k. We show that either x_k can be obtained from x_{k-1} by a legal firing, or $x_k = x_{k-1}$. Indeed, since x_{k-1} is in the chamber C_{k-1}, the amplitude of i_k must be nonpositive. If it is negative, we may fire i_k; if it is zero, then x_{k-1} is fixed by firing i_k, so $x_{k-1} = x_k$.

Thus, we can travel through the x_k's by performing the firing s_{i_k} when it is legal and omitting it when it is not (and hence i_k has zero amplitude). We finally arrive at $x_\ell = w(x)$. □

Thus, it is enough to show that, for any two chambers C and C', there is a point in the interior of C which can by taken to C' by a legal firing sequence. Let ρ be the unique vector of $\delta^\perp$ which is 1 on every vertex of Γ_0. If i is any extending vertex of Γ, we write ρ^i for the analogous construction with respect to that vertex. Our preferred vector in the interior of a given chamber will be the unique vector of the form $w(\rho)$ for $w \in W_0$. We remark that, since all ρ^i are in the same Weyl orbit, this preferred point is also the unique vector of the form $w'(\rho^i)$ for any other extending vertex i.

Furthermore, one can always play the numbers game on Γ_0 until all the amplitudes on Γ_0 are nonnegative; this will carry one into the *dominant* Weyl chamber for W_0. Similarly, by playing the numbers game in reverse, one sees that every configuration is obtainable from one in the *antidominant* Weyl chamber for W_0.

Thus, we will focus on how to get to the antidominant chamber (and, equivalently, how to get from the dominant chamber). More precisely, we will focus on how to get to a chamber which is antidominant for some choice of extending vertex, not necessarily the original one we fixed for reference purposes. By playing the numbers game in reverse, this also shows how to get to a fixed chamber from a chamber which is dominant for some choice of extending vertex.

It turns out that the optimal strategy to get from a point in C of the form $w(\rho)$ (for $w \in W_0$) to an antidominant chamber for some extending vertex is to only fire vertices whose amplitude is less than -1:

Theorem 3.3. *For every $w \in W_0$, starting with $w(\rho)$ and playing the numbers game by arbitrarily firing vertices of amplitude less than -1, one obtains $-\rho^i$ for some extending vertex $i \in \Gamma$. Regardless which moves are chosen, the total number of moves is the same, as is the vertex i. This is the minimum number of moves required to get from $w(\rho)$ to a configuration of the form $-\rho^{i'}$, and all minimum-length firing sequences are of this form (with $i = i'$).*

Remark 3.4. One can deduce from the proof below that, moreover, the *score vector* is the same regardless of the moves chosen: this means the configuration vector in $\mathbb{R}^I$ which records, at each vertex, the sum of negative all the amplitudes fired at that vertex in the course of playing the game.

Using Theorem 3.3, we make the following definition: for every extending vertex i, let $\iota(i)$ be the extending vertex such that, starting at ρ^i and firing vertices of amplitude less than -1, one arrives at $-\rho^{\iota(i)}$. Note that starting at $\rho^{\iota(i)}$ and firing vertices of amplitude less than -1 reverses this path, ending at $-\rho^i$. Thus, ι is an involution.

To prove the theorem, we will use two lemmas:

Lemma 3.5. *Let $\sigma \in \mathbb{Z}^I$ have the following properties:*

(i) *σ is in $\delta^\perp$.*
(ii) *For each $i \in I$, $\sigma_i \geq -1$.*
(iii) *For every root $\alpha \in \Delta$, $\alpha \cdot \sigma \neq 0$.*

Then $\sigma = -\rho^i$ for some extending vertex i.

Note in particular that, for every $w \in W_0$, the vector $w(\rho)$ satisfies (i) and (iii).

Figure 1 demonstrates Lemma 3.5 for the group $\tilde{A}_2$. The dots depict the lattice $\mathbb{Z}^I \cap \delta^\perp$, the grey triangle is the region of $\delta^\perp$ satisfying condition (ii), and the three solid lines are the hyperplanes on which (iii) does *not* hold. Note that there are only three dots which are in the triangle and not on the solid lines; these are the configurations of the form $-\rho^i$, for i an extending vertices of Γ.

Proof of Lemma 3.5. Recall that Δ^0 denotes the root system of W_0. Let I_0 denote the set of vertices of the associated subgraph $\Gamma_0 \subset \Gamma$, and let j be the single (extending) vertex in $I \setminus I_0$. The simple roots of Δ are the simple roots α^i for $i \in I_0$, together with $\alpha^j := \delta - \beta_{\text{long}}$, where β_{long} is the longest root in Δ^0. Using (i) and (ii), we deduce that $\gamma \cdot \sigma \geq -1$ for γ in $R := \{\alpha^i\}_{i \in I_0} \cup \{-\beta_{\text{long}}\}$.

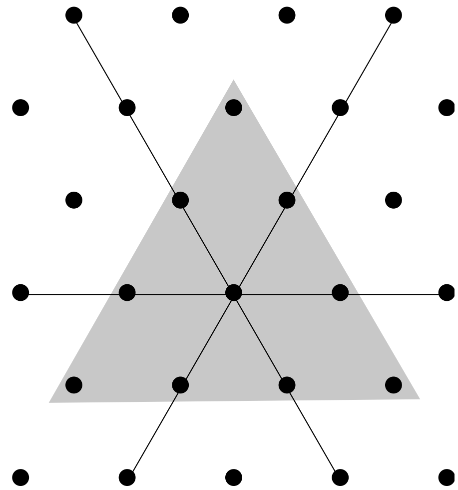

FIGURE 1. Lemma 3.5

Taking inner product with σ yields a linear function on Δ^0, which, by condition (iii), is not zero on any root. Let $\mathcal{N}$ be the subset of Δ^0 whose inner product with σ is negative; so $\mathcal{N} = -w(\Delta^0_+)$ for some $w \in W_0$. For any $\gamma \in \mathcal{N}$, $\gamma = -\sum_{i \in I_0} b_i w(\alpha^i)$ for some nonnegative integers b_i, and then $\gamma \cdot \sigma \leq -\sum b_i$. If γ is an element of $\mathcal{N}$ which is not of the form $-w(\alpha^i)$ then we deduce that $\gamma \cdot \sigma < -1$, and thus $\gamma \notin R$.

Combining the observations of the last two paragraphs, we see that $R \subseteq w(\Delta^0_+) \cup \{-w(\alpha^i)\}_{i \in I_0}$. Suppose, for the sake of contradiction, that $-w(\alpha^i) \notin R$ for some $i \in I_0$. Then $w^{-1}(R)$ lies in the closed half-space whose boundary is spanned by the simple roots other than α^i. But, there is a positive linear combination of all elements of R which is zero. This would imply that R lies in the boundary of this closed half-space, which is impossible since the real span of R is all of $\mathbb{R}^I$.

We deduce that $R \supset \{-w(\alpha^i)\}_{i \in I_0}$. So $\beta \cdot \sigma$ is -1 for all but one element $\beta \in R$. Viewing σ as an element of $\mathbb{R}^I$, this says that all but one coordinate is -1. Let that one coordinate be i. Consider the simple roots of W associated to $I \setminus \{i\}$; modulo δ, these roots form a simple root system for W_0 (namely, $\{-w(\alpha^i)\}_{i \in I_0}$). So, i is an extending vertex and $\sigma = -\rho^i$. $\square$

Lemma 3.6. *Starting with any configuration v, suppose that is possible to fire r vertices $i_1, i_2, \ldots, i_r$ (counted with multiplicity) of amplitude less than -1 until there are none left. Consider any other sequence of firing s vertices of amplitude less than -1. Then $s \leq r$ and this sequence can be extended to a sequence of r firings of vertices of amplitude less than -1 which terminates at the same configuration. Moreover, the Weyl group element $s_{i_r} s_{i_{r-1}} \cdots s_{i_1} \in W$ is independent of the choice of firing sequence.*

Proof of Lemma 3.6. This is similar to the proof of strong convergence in the numbers game, so we will be brief. Our proof is by induction on r. Notice that, if $v_k < -1$ and $v_\ell < -1$ then the ℓ coordinate will still be less than -1 after the k-vertex is fired. Thus, if $r = 1$ then only one coordinate of v is less than -1 and the claim is obvious.

For $r \geq 2$, let $v \to_k \sigma \to \cdots \to \zeta$ be our sequence of length r, beginning by firing k, and let $v \to_\ell \tau \to \cdots \to \zeta'$ be the sequence of length s, beginning by firing ℓ. Then v_k and v_ℓ are both less than -1. Consider alternately firing k and ℓ until both coordinates are ≥ -1. (If this never occurs, then no path from v can terminate.) Let π be the configuration when both coordinates become ≥ -1. We claim that (i) this configuration does not depend on whether we fire k or ℓ first, (ii) the two paths $v \to \sigma \to \cdots \to \pi$ and $v \to \tau \to \cdots \to \pi$ have the same length $t = n_{k\ell}$. Note first that, by the Coxeter relations, the alternating products $s_k s_\ell s_k \cdots \in W$ and $s_\ell s_k s_\ell \cdots \in W$, each of length $n_{k\ell}$, are equal, and take (v_k, v_ℓ) to $(-v_k, -v_\ell)$ if $n_{k\ell}$ is even, and to $(-v_\ell, -v_k)$ if $n_{k\ell}$ is odd. Thus, to prove (i) and (ii), it suffices to show that, if we alternately fire vertices k and ℓ exactly $n_{k\ell}$ times, beginning with either vertex, then only vertices of amplitude less than -1 will be fired. First, since the configurations $(-v_k, -v_\ell)$ or $(-v_\ell, -v_k)$ are dominant restricted to the subgraph on vertices k and ℓ, this must be the result of playing the usual numbers game on this subgraph. By Proposition 2.1, all the amplitudes that are fired are of the form $\alpha \cdot (v_k, v_\ell)$ where α is a positive root for the restriction of the diagram to vertices k and ℓ. Therefore, α is a vector with nonnegative integral entries, implying that the dot product is indeed less than -1. This proves the desired results.

By induction, we can extend $\sigma \to \cdots \to \pi$ to a path $\sigma \to \cdots \to \pi \to \cdots \to \zeta$ of length $r-1$, firing only vertices of amplitude less than -1. Tacking the second part of this path onto the path $\tau \to \cdots \to \pi$, we obtain a path $\tau \to \cdots \to \zeta$ of length $r-1$, firing only vertices of amplitude less than -1. Using induction again, we can complete our previous sequence of length $s-1$, $\tau \to \cdots \to \zeta'$, to another path $\tau \to \cdots \to \zeta' \to \cdots \to \zeta$ of length $r-1$, firing only vertices of amplitude less than -1. Tacking $v \to_\ell \tau$ on the beginning of this path, we obtain an extension of our path $v \to_\ell \tau \to \cdots \to \zeta'$ to a path of length r which terminates at the same configuration ζ as our original path $v \to_k \sigma \to \cdots \to \zeta$, firing only vertices of amplitude less than -1. This proves the desired result.

It remains to show that the product of simple reflections corresponding to the two firing sequences of length r from v to ζ are the same element of W. By induction, the products of the reflections for the parts $\pi \to \cdots \to \zeta$ are the same, so it suffices to show that the products of the reflections for the parts $\sigma \to \cdots \to \pi$ are the same. But these are both alternating products of s_k and s_ℓ of length n_{ij}, so they are the same element of W (as also observed above). This completes the proof. □

Note that there is nothing in the above lemma that requires the number -1: the same is true for any negative number, and neither the number nor v need be integral.

Proof of Theorem 3.3. We begin by showing that there is a path from ρ to some $-\rho^i$ by firing only vertices of amplitude less than -1. Start at ρ and fire vertices of amplitude less than -1 in any manner. Since the W orbit of ρ is finite, either we will reach a configuration with no vertices of amplitude less than -1, or we will repeat a configuration. In the former case, by Lemma 3.5, we are done. In the latter case, we have a path of the form $\rho \to \cdots \to \sigma \to \cdots \to \sigma$. Reversing this path and negating all the configurations, we obtain a path $-\sigma \to \cdots \to -\sigma \to \cdots \to -\rho$ which only fires vertices of amplitude less than -1. So we have two paths of different lengths from $-\sigma$ to $-\rho$, contradicting Lemma 3.6. Note that we now have enough to prove Theorem 3.1.

We continue with the proof of Theorem 3.3. Let σ be of the form $w(\rho)$ for $w \in W_0$. Because the numbers game for W_0 terminates, it is possible to get from σ to ρ by firing only vertices of negative amplitude. By the result we just established, it is possible to get from σ to some $-\rho^i$ by firing only vertices of negative amplitude. Let the length of a shortest path from σ to some $-\rho^i$ be m; we must show that this path involves only firing vertices of amplitude less than -1. Our proof is by induction on m; the base case $m = 0$ is obvious. If σ is of the form $-\rho^i$, we are done. If not then, by Lemma 3.5, $\sigma_k < -1$ for some k.

Let the first step of our path go from $\sigma \to \sigma'$, firing ℓ. The rest of the path, from σ' to $-\rho^i$, must be a shortest path from σ' to any $-\rho^{i'}$ and hence, by induction, must only involve firing vertices of amplitude less than -1. Let $\sigma_\ell = a$. We need to show that $a < -1$.

Starting at σ, alternately fire k and ℓ until the k and ℓ coordinates are both positive. The resulting configuration, τ, is the same whether we fire k or ℓ first, and the length of the resulting path from σ to τ is $n_{k\ell}$ in either case (see the proof of Lemma 3.6). Enroute from σ' to τ, all the vertices fired are of amplitude < -1. Using Lemma 3.6, we can fire $m - n_{k\ell}$ more vertices of amplitude less than -1 to get from τ to $-\rho^i$. Now, consider the path $\sigma \to \cdots \to \tau' \to \tau \to \cdots \to (-\rho^i)$ which starts by firing k. Then the vertex which is fired when going from τ' to τ has amplitude a. The path from τ' to $-\rho^i$ must be shortest possible to any $-\rho^{i'}$ (otherwise there would be a shorter path from σ to some $-\rho^{i'}$). So, by induction, every vertex which is fired in this path has amplitude less than -1. In particular, $a < -1$, as desired. $\square$

Remark 3.7. One may explicitly determine the involution ι appearing above:

Proposition 3.8. *The involution ι is the restriction of a unique automorphism of Γ. It is trivial for exactly the graphs*

$$\widetilde{A_{2m}}, \widetilde{B_{4m-1}}, \widetilde{B_{4m}}, \widetilde{D_{4m}}, \widetilde{D_{4m+1}}, \widetilde{E_6}, \widetilde{E_8}, \widetilde{F_4}, \widetilde{G_2}, \quad m \geq 1. \tag{3.9}$$

For graphs $\widetilde{A_{2m-1}}$, ι is the involution sending every vertex in Γ to its antipodal vertex. For type $\widetilde{D_{4m+2}}, \widetilde{D_{4m+3}}$, ι interchanges extending vertices which are adjacent to a common vertex. For types $\widetilde{B_{4m+1}}, \widetilde{B_{4m+2}}, \widetilde{C_{m+1}}, \widetilde{E_7}$, ι is the restriction of the unique nontrivial automorphism of the graph.

We will sketch a proof of the proposition in Remark 5.6.

4. The resulting poset

At this point, we have proved that, for any point v in the boundary of the Tits cone, it is possible to get from v to any other point in the W_0 orbit of v. Moreover, we have described the most efficient way to get to an antidominant chamber (for some choice of extending vertex). Theorem 3.3 suggests defining a directed graph structure as follows: The vertices are elements of $W_0(\rho)$, and there is an edge from u to v if one can go from u to v by firing a vertex of amplitude less than -1. Our Theorem 3.3 states this graph is acyclic and that each connected component contains a unique source, ρ^i, and a unique sink, $-\rho^{\iota(i)}$. If we take the transitive closure of this graph, we will obtain a graded poset, of which this graph is the Hasse diagram. We are thus naturally lead to partition $W_0(\rho)$ (and hence also W_0) into a collection of graded posets, one for each extending vertex of Γ. Write P^i for the poset of configurations obtainable from ρ^i by firing vertices of amplitude less than -1. We will spend the rest of this paper studying these posets.

By Theorem 3.3,

$$W(\rho) = \{w(\rho) : w \in W_0\} = \bigsqcup_{i \text{ an extending vertex}} P^i. \tag{4.1}$$

Furthermore, each P^i is a graded poset: this means that each element $v \in P^i$ has a well-defined degree, given by the number of firings of vertices with amplitude less than -1 needed to go from ρ^i to v. They are also *self-dual*, which means that the poset is isomorphic to the one where the ordering is reversed.

The P^i are all isomorphic posets. Since $W_0 \subset W$ acts freely on $\rho = \rho^j$, we may view (4.1) as a decomposition of W_0 itself into isomorphic graded posets, $W_0 = \bigsqcup_i W_0^i$. Moreover, the isomorphism $W_0^j \xrightarrow{\sim} W_0^i$ is nothing but $w \mapsto r^i w$, where $r^i \in W_0$ is the element such that $r^i \rho = \rho^i$.

We remark that (4.1) is quite canonical. In particular, it does not depend on the choice of the dominant vector ρ: any element u whose restriction to Γ_0 is in the interior of the dominant Weyl chamber (i.e., all amplitudes are positive) gives rise to the same decomposition (4.1), except that P^i are now defined as the graded posets of configurations obtainable along a minimal-length firing sequence from $r^i(u)$ to $w(u)$, where $w \in W$ satisfies $w(\rho) = -\rho^{\iota(j)}$. When we pass to the decomposition of W_0 itself into isomorphic graded posets, the result is independent of u.

In the next sections, we will identify each P_i with an interval under the (left) weak order in the affine Weyl group, and equivalently with the dual of the triangulation of a unit hypercube in the reflection representation. The key technical ingredient, which comes first, is to compare playing the numbers game beginning with ρ_i to the numbers game on a modified configuration which terminates.

5. Relating the numbers game at the boundary to the interior of the Tits cone

From now on, fix an extending vertex j, so $\rho = \rho^j$. The main result of this section is to relate the procedure of starting at ρ^j and firing only amplitudes less than -1 to the procedure of starting within the Tits cone at certain points and playing the numbers game arbitrarily.

Let $I_0 := I \setminus \{j\}$. Let $\beta_{\text{long}} \in (\Delta_0)_+$ be the longest root of $(\Delta_0)_+$, and let $s_{\beta_{\text{long}}} \in W_0$ be the corresponding reflection. Let L be the length function on W and let b be a real number in the interval $(1, 1 + \frac{1}{L(s_{\beta_{\text{long}}})-1})$. With these definitions, let $u_b = \rho^j + b\omega^j$, where ω^j is the configuration which is 1 at the vertex j and zero elsewhere.

The main result of this section is

Proposition 5.1. *A sequence of vertices $i_1, i_2, \ldots$ is a valid firing sequence for the numbers game starting with u_b if and only if it is a firing sequence of amplitudes less than -1 starting with ρ^j. The final configuration reached under a maximal such firing sequence (starting with u_b) is $(b-1)\rho^{\iota(j)} + b\omega^{\iota(j)}$ for the involution ι introduced above.*

Note that the proposition essentially requires b to be as specified. For values $b < 1$, the set of firing sequences starting with ρ^j of amplitudes less than -1 is a proper subset of the set of valid firing sequences starting with u_b; for values $b \geq 1 + \frac{1}{L(s_{\beta_{\text{long}}})-1}$, the set of firing sequences starting with ρ^j of amplitudes less than -1 properly contains the set of valid firing sequences starting with u_b. (We remark that the proposition still holds if b equals 1, but since this places u_b on the boundary of its Weyl chamber, we avoid this possibility.)

The proof of the proposition relies on the following basic lemma. For each vertex $k \in I_0$, let $T_k : \mathbb{R}^I \to \mathbb{R}^I$ be the "translation" element of the form

$$T_k(v) = v + (\delta \cdot v)(\omega^k - \delta_i \omega^j). \tag{5.2}$$

Let $P_0^\vee$ be the coweight lattice for Δ^0 (with basis the fundamental coweights $\omega^i, i \in I_0$) and let $Q_0^\vee := \langle \alpha^\vee : \alpha \in \Delta_0 \rangle \subset P_0^\vee$ be the coroot sublattice. In the basis of fundamental coweights, $P_0^\vee = \mathbb{Z}^{I_0} \supseteq \langle \alpha^\vee : \alpha \in \Delta_0 \rangle$, where $(\alpha^\vee)_i = \langle \alpha, \alpha^i \rangle$ for all $i \in I_0$.

Lemma 5.3. *For every vertex $k \in I_0$, there is a unique element $t_k \in W$ whose action on $\mathbb{R}^I$ is of the form $t_k = T_k \circ \gamma_k$, where $\gamma_k : \mathbb{R}^I \to \mathbb{R}^I$ is a permutation of coordinates corresponding to an automorphism of the graph Γ. The map $\omega^k \mapsto \gamma_k$ induces a injective homomorphism from the group[4] $P_0^\vee / Q_0^\vee$ into* $\text{Aut}(\Gamma)$.

Proof. Fix $b > 0$ and consider the hyperplane $\mathcal{H}_b := \{v \in \mathbb{R}^I \mid \delta \cdot v = b\}$, fixed under W. It suffices to show that the lemma holds restricted to $\mathcal{H}_b$. Note that the triangulation of $\mathcal{H}_b$ by its intersection with the Weyl chambers has the translational symmetry $T_k|_{\mathcal{H}_b}$. Thus, there must exist a unique element $t_k \in W$ such that t_k takes the dominant Weyl chamber (i.e., the one whose amplitudes are all nonnegative) to its translate under T_k. Therefore, $T_k^{-1} \circ t_k$ must be an isometry of the dominant Weyl chamber. Thus, $T_k^{-1} \circ t_k$ is induced by an automorphism $\gamma_k \in \Gamma$.

To see that this induces a homomorphism $\psi : P_0^\vee \to \text{Aut}(\Gamma)$, we need to show that the γ_k all commute with each other. When Γ is not of type A or D, this is immediate since $\text{Aut}(\Gamma)$ is abelian. In the case of type A, a straightforward computation shows that, when k is adjacent to j, then γ_k is a rotation of the diagram Γ (moving each vertex to an adjacent one), and all other γ_ℓ can be obtained as products of conjugates of this one. Therefore, the images of all the γ_ℓ generate the abelian normal subgroup $\mathbb{Z}/n < D_{2n} = \text{Aut}(\Gamma)$, in this case. In the case of type D, the image of P_0

[4]The group $P_0^\vee / Q_0^\vee$ is called the *fundamental group* of the root system Δ^0, and it is a standard object.

is a 4-element subgroup of D_4 and is hence abelian (we can also compute explicitly that the image is abelian, like in type A).

Finally, we need to show that the kernel of ψ is $Q_0^\vee$. Let us define

$$s'_k := \begin{cases} s_k, & \text{if } k \in I_0, \\ s_{\beta_{\text{long}}}, & \text{if } k = j. \end{cases} \tag{5.4}$$

Then, $s_{k_1}s_{k_2}\cdots s_{k_m}$ is a translation if and only if $s'_{k_1}s'_{k_2}\cdots s'_{k_m} = 1$. In this case, it equals

$$s_{k_1}s_{k_2}\cdots s_{k_m}(s'_{k_1}s'_{k_2}\cdots s'_{k_m})^{-1},$$

which can be written as a product of conjugates of the translation $s'_j s_j = \psi(\beta^\vee_{\text{long}}) = \prod_{k\in I_0} T_i^{\langle \alpha^k, \beta^\vee_{\text{long}}\rangle}$. In other words, the translations of W are exactly $\psi(Q_0^\vee)$. □

Proof of Proposition 5.1. Consider any valid firing sequence starting from u_b. We first show that the same firing sequence, applied to ρ^j, only involves firing vertices of amplitude less than -1. By Proposition 2.1, each amplitude of the original sequence has the form $\beta \cdot u_b$ for some positive root β, and applied to ρ^j, the amplitude fired is $\beta\cdot\rho^j$. We need to show that $\beta\cdot\rho^j < -1$. More generally, we show this assuming only that β is positive and $\beta \cdot u_b$ is negative.

Note that $\beta\cdot u_b = \beta\cdot\rho^j + b\beta\cdot\omega^j$. We claim that $\beta\cdot\omega^j \geq 1$. Indeed, otherwise β is a positive root supported away from j (i.e., on I_0), and this would imply $\beta \cdot u_b = \beta\cdot\rho^j > 0$, since ρ^j is dominant on I_0. This is a contradiction, so $\beta\cdot\omega^j \geq 1$.

By hypothesis, $b > 1$ and $\beta\cdot u_b < 0$. So, $\beta\cdot\rho^j = \beta\cdot u_b - b\beta\cdot\omega^j < -1$, as desired. So a valid firing sequence for u_b gives a firing sequence for ρ^j where are amplitudes fired are less than -1.

We next show that, starting at u_b, the numbers game terminates at a configuration of the form $y_b^i := (b-1)\rho^i + b\omega^i$, for some extending vertex i (we will show that $i = \iota(j)$ later). The numbers game for u_b must terminate in finitely many moves, by Theorem 1.2.(iii). The terminal configuration is the unique dominant configuration in the Weyl orbit of u_b.

We claim that, by the assumption on b, the configuration y_b^i is dominant. First, since $b > 1$, $(y_b^i)_\ell > 0$ for $\ell \neq i$. Next, $(\rho^i)_i = -L(s_{\beta_{\text{long}}})$, and hence $(y_b^i)_i = (1-b)L(s_{\beta_{\text{long}}}) + b$, which is positive iff $b < 1 + \frac{1}{L(s_{\beta_{\text{long}}})-1}$. Thus, y_b^i is dominant. Moreover, y_b^i is positive, i.e., in the interior of the dominant chamber. Hence, it suffices to show that y_b^i is in the Weyl orbit of u_b, for some extending vertex i.

Recall the translations T_k of (5.2). Let us denote these by T_k^j; for any extending vertex i, denote the corresponding translations by T_k^i. Fix an extending vertex i. By Lemma 5.3, for all $k \neq i$, there exists an automorphism γ_k^i of the extended Dynkin diagram Γ such that $t_k^i = T_k^i\gamma_k^i \in W$. Note that $\prod_{k\neq i} T_k^i(y_b^i) = -\rho^i + b\omega^i$. We can write $\prod_{k\neq i} T_k^i = \gamma w$ where γ is an automorphism of Γ and w is in W. So the W-orbit of y_b^i contains $\gamma^{-1}(-\rho^i + b\omega^i) = -\rho^{\gamma^{-1}(i)} + b\omega^{\gamma^{-1}(i)}$. Since γ is an automorphism of Γ, the vertex $i' := \gamma^{-1}(i)$ is an extending vertex. Moreover, the map $i \mapsto i'$ must be a bijection, since we can apply an automorphism of Γ to everything, and the y_b^i are all in distinct Weyl orbits as they are distinct dominant vectors. Hence, there exists a choice of i such that $i' = j$. Let i be this choice. Thus, y_b^i is in the W-orbit of $-\rho^j + b\omega^j$.

By playing the numbers game on I_0, the configuration $-\rho^j + b\omega^j$ can be taken to $u_b = \rho^j + b\omega^j$. As a result, u_b is in the Weyl orbit of y_b^i. Since we also showed that y_b^i is in the interior of the dominant Weyl chamber, u_b is in the interior of its Weyl chamber, and the element $w \in W$ constructed such that $w(u_b) = y_b^i$ is unique.

Next, take any full firing sequence $u_b \to \cdots \to y_b^i$. We now show that the same firing sequence takes ρ^j to $-\rho^i$. Since we already proved that this is a firing sequence of vertices of amplitudes less than -1, it will then follow from Theorem 3.3 that $i = \iota(j)$.

To prove the claim, apply the firing sequence instead to $u_x = \rho^j + x\omega^j$, for x an indeterminate. By construction of the element $w \in W$ which takes u_b to $-\rho^i + b\omega^i$ above, we see that the same element takes u_x to $-\rho^i + x\omega^i$. Since w is the unique such element, the firing sequence itself must take u_x to $-\rho^i + x\omega^i$. Setting $x = 0$, we see that it takes ρ^j to $-\rho^i$, as desired.

We are now ready to prove that any firing sequence from ρ^j of vertices of amplitude less than -1 is also a valid firing sequence for u_b. It suffices to prove the result for a full firing sequence, i.e., one of the form $\rho^j \to \cdots \to -\rho^{\iota(j)}$. By Proposition 2.1, the corresponding product of simple reflections is reduced. By Lemma 3.6, the resulting element $w \in W$ is independent of the sequence. Now, if we apply this sequence of firings to u_b, the result $w(u_b)$ must be $y_b^{\iota(j)}$, since it is true for at least one sequence, namely one obtained from a valid firing sequence $u_b \to \cdots \to y_b^{\iota(j)}$. Moreover, the lengths of the sequences must be the same.

To conclude, we claim that, if $u \to \cdots \to v$ is any valid firing sequence with v dominant (i.e., a full firing sequence), then any other firing sequence from u to v of the same length is also valid. Applied to our original sequence with $u = u_b$ and $v = y_b^{\iota(j)}$, this yields the desired result.

To prove the claim, recall the following construction, introduced earlier in [GS09, (3.5)]: Given u, consider the set $X_u := \{\alpha \in \Delta_+ \mid \alpha \cdot u < 0\}$. Each valid move of the numbers game decreases the size of this set by one. On the other hand, if we fire any vertex with nonnegative amplitude, the set does not decrease in size. Hence, the valid firing sequences $u \to \cdots \to v$ (when they exist) are exactly the firing sequences of minimal length. This proves the claim, and hence the proposition. □

Remark 5.5. In view of Proposition 2.1, the claim occupying the last two paragraphs of the proof can be reformulated as follows (in the case where u is in the interior of its Weyl chamber, as is our situation): A firing sequence from an arbitrary configuration u to a configuration $w(u) \in \mathbb{R}^I_{>0}$, for $w \in W$, is valid if and only if the length of the sequence is $L(w)$. See also Lemma 6.1 and its proof.

Note that the number of firing steps to get from u_b to $(b-1)\rho^{\iota(j)} + b\omega^{\iota(j)}$ is the number of reflecting hyperplanes separating u_b from the fundamental domain. In type $\tilde{A}_n$, we can compute this directly as follows. Let $\alpha \in \mathbb{Z}^I_{\geq 0}$ be such a positive root, and let $m := \alpha_j$. Then $\alpha = (m-1)\delta + \alpha'$, where α' is a positive root supported on a segment containing j, of length $\leq n - m$. Conversely, any $m \geq 1$ and α' as above uniquely determine a positive root $\alpha \in \mathbb{Z}^I_{\geq 0}$ separating u_b from the fundamental domain. Let us identify $I \cong \{0, 1, 2, \ldots, n\}$, with $j = 0$, and with two integers adjacent if they differ by one modulo $n+1$. Then, to pick a pair of $\alpha' = \alpha^k + \alpha^{k+1} + \cdots + \alpha^n + \alpha^0 + \alpha^1 + \cdots + \alpha^\ell$ and the integer $1 \leq m \leq n - (n - k + \ell + 2)$ is equivalent to picking the triple $\ell < \ell + m < k - 1$ of distinct integers in $\{0, 1, 2, \ldots, n\}$ (where we take $k-1$ modulo $n+1$, i.e., set $k-1$ to be n if $k = 0$). Thus, the total number of such α is $\binom{n+1}{3}$.

We will prove a more general result (and for any extended Dynkin graph) in §8 below.

Remark 5.6. The results of this section allow one to prove Proposition 3.8. We briefly sketch the proof. The first statement, that ι extends to a unique automorphism of Γ, is immediate from the explicit formula for ι. Alternatively, note that ι must commute with all permutations of the extending vertices obtained from automorphisms of Γ which preserve C, and this already implies the first statement. In fact, it narrows the possibilities for ι down to at most two choices for each graph: either the trivial automorphism, or the unique nontrivial involution in the center of $\mathrm{Aut}(\Gamma)$ in the cases $\widetilde{A_{2m-1}}, \widetilde{B_m}, \widetilde{C_m}, \widetilde{D_m}$, and $\widetilde{E_7}$. So one can actually restrict to the latter cases (and, using folding, i.e., viewing configurations in $\widetilde{C}$ as configurations in $\widetilde{A}$ which are symmetric under the antipodal map, these cases are equivalent, and similarly the type $\widetilde{B}$ and $\widetilde{D}$ cases are equivalent).

To prove the stated formula, it suffices to compute the map $i \mapsto i'$ from the proof of Proposition 5.1: since this sends $\iota(j)$ to j for all j, and ι is an involution, this map is exactly ι. We can do this on a case-by-case basis as follows. Given an extending vertex i, it suffices to compute the element $\gamma \in \mathrm{Aut}(\Gamma)$ such that $\prod_{k \neq i} T^i_k = \gamma w$ where γ is an automorphism of Γ and w is in W. To do so, first we compute explicitly the elements $\gamma^i_k \in \mathrm{Aut}(\Gamma)$ such that $t^i_k = T^i_k \gamma^i_k$ is in W, which is straightforward. Then it is straightforward to compute γ from this and hence $i' = \iota(i)$. Note that, by the preceding paragraph, it suffices to do this for a single extending vertex i, and only in types $\widetilde{A_{2m-1}}, \widetilde{D_m}$, and $\widetilde{E_7}$ (alternatively, one can replace type $\widetilde{A}$ by $\widetilde{C}$ and/or type $\widetilde{D}$ by $\widetilde{B}$).

6. Identifying P_i with an interval in the weak order

As we see, the graded poset W^j_0 is quite different from the poset corresponding to W_0 under the weak or Bruhat orders: rather than having at most quadratic degree in the number of vertices of the Dynkin diagram, the top degree of W^j_0 is *cubic* in the number of vertices, at least in types A (we will see later on that the same is true in all types).

Using Proposition 5.1, we can identify the poset W^j_0 with an interval under the (left) weak order in the affine Weyl group. First, note that, for every element $v \in P^j \cong W^j_0$ obtained from ρ by a sequence of firings $i_1, i_2, \ldots, i_m \in I$ of amplitudes less than -1, Lemma 3.6 shows that the element $s_{i_m} s_{i_{m-1}} \cdots s_{i_1} \in W$ depends only on v and not on the choice of firing sequence. That is, we obtain an embedding $P^j \hookrightarrow W$. Let $\varphi : W^j_0 \cong P^j \hookrightarrow W$ be the resulting composition. This is a section of the quotient $\chi : W \twoheadrightarrow W_0$ defined by $w(v) = \chi(w)(v)$ for any $v \in \mathbb{R}^{I_0}$, i.e., $\chi(s_i) = s_i$ for $i \in I_0$, and $\chi(s_j) = s_{\beta_{\mathrm{long}}}$ is the reflection about the maximal root β_{long} of Δ_0. By φ being a section of χ, we mean precisely that the composition $\chi \circ \varphi : W^j_0 \to W_0$ equals the inclusion map. We will now show that the image poset $\varphi(W^j_0) \subset W$ is nothing but an interval in W under the weak order.

Recall first the definition of weak order. For $g \in W$, the *length* of g, denoted $L(g)$, is the minimal number of simple reflections s_i needed to multiply to g. The *left* weak order in W is the ordering such that $g \leq_l h$ if and only if $L(h) = L(g) + L(hg^{-1})$, and the *right* weak order in W is the ordering such that $g \leq_r h$ if and only if $L(h) = L(g) + L(g^{-1}h)$.

Generally, define the *numbers game ordering* by: u is less than v if v can be obtained from u by playing the numbers game.

Lemma 6.1. *Let Γ be any graph associated to a Coxeter group W. Given any configuration u for which the numbers game terminates at $v = w(u)$ for $w \in W$, the map $W \to \mathbb{R}^I, g \mapsto g(u)$ restricts to an isomorphism of the interval $[\mathrm{id}, w]_{<_l}$ with the numbers game poset from v to u.*

Proof. By strong convergence, if a valid firing takes $g(u)$ to $s_i g(u)$, where $g \in W$ and u is dominant, then $L(s_i g) = L(g) - 1$. As a consequence, it inductively follows that it takes exactly $L(g)$ moves to take $g(u)$ to u by playing the numbers game. The result follows immediately. ☐

In our situation, let w^{top} be the element of W such that $w^{\mathrm{top}}(u_b)$ is dominant. Proposition 5.1 tells us that the poset P^i is isomorphic to the numbers game poset from u_b to $w^{\mathrm{top}}(u_b)$. This is nothing but the interval $[e, w^{\mathrm{top}}]$ in the left weak order. To summarize:

Corollary 6.2. *The isomorphism φ takes the poset W^j_0 to the interval $[1, \varphi(w^{top})]_{<_l}$.*

In the next section, we will recall the geometric way to think of this interval.

7. Triangulation of the unit hypercube in the reflection representation

Denote by $\mathcal{K}_b := \{v \in \mathcal{H}_b \mid v_i \in [0, b], \forall i \in I_0\}$ the "unit hypercube" in the hyperplane $\mathcal{H}_b := \{v \in \mathbb{R}^I \mid \delta \cdot v = b\}$. Note that $\mathcal{K}_b$ is a fundamental domain under the group generated by the translations T_j used in Lemma 5.3, and its image under $\mathcal{H}_b \xrightarrow{\sim} \mathbb{R}^{I_0}$ is the hypercube $[0, b]^{I_0}$.

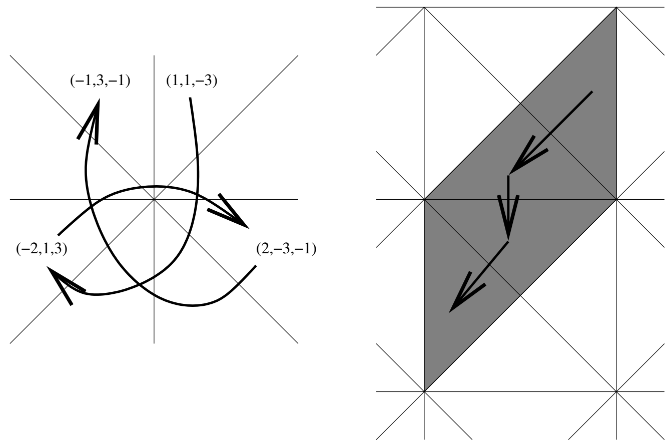

FIGURE 2. The graph $\Gamma(W_0^j)$ in B_2, and the unit hypercube in $\tilde{B}_2$

Let us associate to the poset W_0^j its Hasse diagram $\Gamma(W_0^j)$, i.e., the graph whose vertices are elements of W_0^j and whose directed edges are $g \to h$ such that $h(\rho)$ is obtained from $g(\rho)$ by firing a single vertex of amplitude less than -1.

Let $D \subset \mathcal{H}_b$ be the dominant Weyl chamber, i.e., $D = \{v \in \mathcal{H}_b \mid v_i \geq 0, \forall i \in I\}$. To any polytope that is the union of Weyl chambers, we associate a dual directed graph, which is the usual dual graph forgetting orientation, with orientation given by $w(D) \to ws_i(D)$ when $L(w) < L(ws_i)$.

Proposition 7.1. *The graph $\Gamma(W_0^j)$ is isomorphic to the dual of the triangulation of the unit hypercube $\mathcal{K}_b$ by Weyl chambers.*

Proof. The dual of the triangulation of $\mathcal{K}_b$ is the interval $[1, w']_{<_r}$ under the right weak order, where w' is the longest element such that $w'D \subset \mathcal{K}_b$. We claim that $w' = (w^{\text{top}})^{-1}$. Given the claim, the result follows immediately from the fact that $[1, w^{\text{top}}]_{<_l}$ is isomorphic to $[1, w']_{<_r}$ under the inversion map (which sends the left weak order to the right weak order).

To prove the claim, first note that the Weyl chamber containing u_b is in $\mathcal{K}_b$ and is the one incident to the corner v of $\mathcal{K}_b$ that is opposite to e, i.e., to the corner v given by $v_i = b$ for $i \in I_0$. (To see that u_b and v are in the same Weyl chamber, one can take b very close to 1 without changing which chamber u_b is in, which would make u_b very close to v.) Thus, $\varphi(w^{\text{top}})^{-1}$ is the longest element which takes D to $\mathcal{K}_b$, i.e., we must have $w' = \varphi(w^{\text{top}})^{-1}$. □

In figure 2, we demonstrate the above concepts in type $\tilde{B}_2$. On the left, we depict the graph $\Gamma(W_0^j)$. The point (a, b, c) means the point of $\mathbb{R}^I$ with those coordinates. Our convention is that the first coordinate corresponds to the root $(1, 0)$ of B_2, the second coordinate to the root $(-1, 1)$ of B_2, and the third coordinate to $(-1, -1)$, the negation of the longest root. On the right, we show the unit hypercube of $\tilde{B}_2$, and how $\Gamma(W_0^j)$ occurs as the dual to this hypercube.

8. THE RANK GENERATING FUNCTION OF W_0^j

8.1. **The general formula.** In what follows, we will consider every subset of W as being endowed with the graded poset structure given by the *right weak order*, $<_r$. Also, for every finite Coxeter group associated to a graph Γ_0 with vertex set I_0, let $m_1, m_2, \ldots, m_{|I_0|}$ be its Coxeter exponents.

Recall that the rank generating function $h(P; t)$ of a graded poset P is defined as

$$h(P; t) = \sum_{d \geq 0} |\{x \in P : |x| = d\}| t^d, \tag{8.1}$$

where $|x|$ denotes the degree of x. The main goal of this section is to explicitly compute the rank generating function of the graded poset W_0^j:

Theorem 8.2. *The rank generating function of P^i is given by*

$$h(P^i;t) = \frac{\prod_{i\in I_0}(1-t^{L(t_i)})}{\prod_{i=1}^{|I_0|}(1-t^{m_i})}. \tag{8.3}$$

Here, the elements $t_i \in W$ are the translations as defined in the Lemma 5.3.

Note that, evaluating the polynomial at $t=1$ and using that W_0 decomposes into isomorphic copies of P^i, one copy for each extending vertex, we obtain (1.3).

We remark that there is always a way to rearrange the factors in the denominator, i.e., to assign to each vertex $i \in I_0$ an exponent m_i, so that $\frac{1-t^{L(t_i)}}{1-t^{m_i}} = 1+t^{m_i}+t^{2m_i}+\cdots+t^{L(t_i)-m_i}$ is a polynomial. In some sense, this can be done uniquely: see §8.2.

Proof. Let $H_+ \subset W$ be the semigroup generated by the elements t_i defined in Lemma 5.3, i.e., the elements of W of the form $T_{i_1}T_{i_2}\cdots T_{i_m}\gamma$, where $i_1,\ldots,i_m \in I_0$, the T_{i_j}'s were defined in 5.2, and $\gamma \in \mathrm{Aut}(\Gamma)$. We claim that

$$W = W_0 H_+ \varphi(W_0^j)^{-1}, \tag{8.4}$$

and moreover that this is a direct product decomposition, i.e., every element $w \in W$ has a unique decomposition as $w = w_0 h \varphi(w')^{-1}$ where $w_0 \in W_0$, $h \in H_+$, and $w' \in W_0^j$. Finally, we claim that this decomposition satisfies

$$L(w) = L(w_0) + L(h) + L(\varphi(w')^{-1}). \tag{8.5}$$

As a consequence, by taking rank generating functions (using the well known formulas [BB05, Theorems 7.1.5, 7.1.10] for $h(W;t)$ and $h(W_0;t)$),

$$h(W;t) = \frac{h(W_0;t)}{\prod_{i=1}^{|I_0|}(1-t^{m_i})} = h(W_0;t)\frac{1}{\prod_{i\in I_0}(1-t^{L(t_i)})}h(W_0^j;t), \tag{8.6}$$

which proves the theorem, subject to proving (8.4) and (8.5), which we do now. Applying both sides of (8.4) to the dominant Weyl chamber in $\mathcal{H}_b$, the claim follows from the statement that a fundamental domain for $W_0 \subset W$ in $\mathcal{H}_b$ is given by the image of the dominant Weyl chamber under $\varphi(W_0^j)^{-1}H_+$, i.e., the cone $\{v \in \mathcal{H}_b \mid v_i \geq 0, \forall i \in I_0\}$. In turn, this statement follows because, under the projection $\mathcal{H}_b \xrightarrow{\sim} \mathbb{R}^{I_0}$ by forgetting the j-coordinate, this cone is the preimage of the dominant W_0-chamber in $\mathbb{R}^{I_0}$. □

8.2. **Explicit formulas.** To explicitly write down $h(W_0^j;t)$, it is enough to write down the lengths $L(t_i)$ and the Coxeter exponents m_i. In Figure 3 we organize this data suggestively, by labeling the vertices of every Dynkin graph Γ_0 by two positive integers: $L(t_i)$, and a Coxeter exponent m_i such that $m_i \mid L(t_i)$, so that each m_i occurs once. The figure uses the following notation: the *odd part*, $\{m\}_2$, of $m \in \mathbb{Z}_+$ is the maximal odd factor of m.

Set $Q_m(t) := \frac{t^m-1}{t-1} = 1+t+t^2+\cdots+t^{m-1}$. Then, as remarked in the previous subsection, Theorem 8.2 implies that

$$h(W_0^j;t) = \prod_{i\in I_0} Q_{\frac{L(t_i)}{m_i}}(t^{m_i}), \tag{8.7}$$

which is a factorization of $h(W_0^j;t)$ by polynomials whose nonzero coefficients are all 1.

The top degree of the poset W_0^j is the degree of the rank generating function. Using Figure 3, we obtain this in Figure 4. For the series of types A and D, these degrees are cubic in the number of vertices; in the exceptional cases, one may find similar identities, such as $120 = 6\cdot5\cdot4, 336 = 8\cdot7\cdot6,$

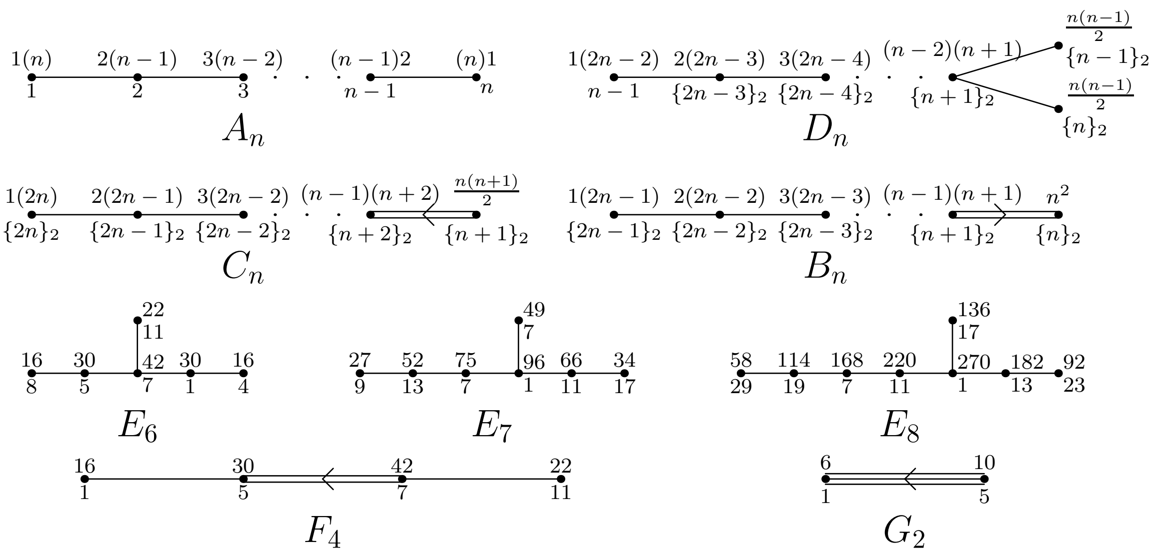

FIGURE 3. The lengths $L(t_i)$ and Coxeter exponents m_i for all Dynkin graphs

A_n	B_n	C_n	D_n	E_6	E_7	E_8	F_4	G_2
$\binom{n+1}{3}$	$\frac{n(n-1)(4n+1)}{6}$	$\frac{n(n-1)(4n+1)}{6}$	$4\binom{n}{3}$	120	336	1120	86	10

.

FIGURE 4. The top degrees of the posets W_0^j

and $1120 = \frac{1}{3}(16 \cdot 15 \cdot 14)$). An upper bound on the number of valid moves in the numbers game required to go from any vector v on the extended Dynkin graph satisfying $\delta \cdot v = 0$ to any element of its Weyl orbit is given by the sum of this degree and the weak order degree of W_0 (the latter being quadratic in the number of vertices), and is therefore cubic in the number of vertices.

Let us consider the question of how unique the assignment of the exponents m_i to the vertices is such that $m_i \mid L(t_i)$. For exceptional types, this assignment of the exponents to the vertices is unique, except in the E_6 case, where it is unique up to swapping m_i with $m_{i'}$ when i, i' are images of each other under an element of $\mathrm{Aut}(\Gamma_0)$. For each infinite series, one can make a uniqueness statement if one views the collection of graphs for all n together. For example, for types A, B, and C, if we label the vertices for such a series subject to the condition that the maximal segment of consecutively-numbered vertices goes to infinity, then this is the unique assignment of m_i so that the m_i are given by a polynomial in n and i or the odd part of such a polynomial (and in fact the vertices must be labeled as above, up to symmetry). For type D, this is true except for the leftmost vertex above, which is assigned a different polynomial, $n - 1$.

We remark that, for types B and C, even though their finite Weyl groups are identical, the rank generating functions $h(W_0^j(B_n); t)$ and $h(W_0^j(C_n); t)$ are not equal, and in particular $W_0^j(B_n) \ncong W_0^j(C_n)$. However, $h(W_0^j(B_n); 1) = h(W_0^j(C_n); 1)$; indeed, both must equal $\frac{1}{2}|W_0|$. Moreover, $\deg h(W_0^j(B_n); t) = \deg h(W_0^j(C_n); t)$.

8.3. **Combinatorial interpretation of the rank generating function for type** A. In this section, we consider the case of $\tilde{A}_{n-1}$. In the proof of Theorem 8.2, we computed the rank generating functions of the subset of W which takes the dominant Weyl chamber in $\mathcal{H}_b$ to the inverse image of the dominant Weyl chamber for A_{n-1}; specifically, we showed that this series is $1/(1-t)(1-$

$t^2)\cdots(1-t^{n-1})$. In [EE98, Section 9.4], Eriksson and Eriksson gave a combinatorial proof of this result. We sketch their proof, and explain how to modify it to give a combinatorial proof of Theorem 8.2. Let here W and W_0 be the affine and finite Weyl groups of types $\widetilde{A_{n-1}}$ and A_{n-1}, respectively. We identify the vertices I of the extended Dynkin diagram Γ with $\{1,2,\ldots,n\}$. Then $\mathbb{R}^I$ is identified with $\mathbb{R}^n$. Let also $D \subseteq \mathbb{R}^n = \mathbb{R}^n_{\geq 0}$ be the dominant Weyl chamber. Let $\mathcal{K}_1 := [0,1]^n \subseteq D$ be the unit hypercube.

Let $\tilde{S}_n$ be the set of permutations $i \mapsto \sigma_i$ of the integers such that $\sigma_{i+n} = \sigma_i + n$ and $\sum_{i=1}^n \sigma_i = \sum_{i=1}^n i$. Define a map $\partial : \tilde{S}_n \to \mathbb{R}^n$ by $(\partial\sigma)_i = \sigma_i - \sigma_{i-1}$; it is well known that this map is injective and its image is $W(1,1,\ldots,1)$. In this way, we produce a bijection $\psi : \tilde{S}_n \xrightarrow{\sim} W$. The preimage under ∂ of the dominant chamber of $\mathbb{R}^n$ consists of those permutations with $\sigma_1 < \sigma_2 < \cdots < \sigma_n$. The preimage of the unit hypercube consists of those permutations where, in addition, $\sigma_1 + n > \sigma_2$, $\sigma_2 + n > \sigma_3$, …, and $\sigma_{n-1} + n > \sigma_n$.

Given $\sigma = (\sigma_i)$ such that $\partial\sigma \in D$, define an n-tuple $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_{n-1})$ as follows. Let i be an integer between 1 and $n-1$. Let U_i be the set of integers t such that $t < \sigma_{i+1}$ and $t \not\equiv \sigma_{i+1}, \sigma_{i+2}, \ldots, \sigma_n \pmod{n}$. Number the elements of U_i as $u_0 > u_1 > u_2 > \cdots$. Then, σ_i is in U_i. Define γ_i by $\sigma_i = u_{\gamma_i}$. If we do this for all i, we obtain $\gamma = (\gamma_1, \ldots, \gamma_{n-1})$.

Remark 8.8. The integer γ_i is the number of times that $n-i$ occurs in the sequence $\delta_\bullet$ constructed in [EE98].

It is straightforward to check that this defines a bijection $\phi : \partial^{-1}(D) \xrightarrow{\sim} \mathbb{Z}^{n-1}_{\geq 0}$ between the preimage under ∂ of the dominant chamber and $\mathbb{Z}^{n-1}_{\geq 0}$. Furthermore, under this bijection, the unit hypercube $\mathcal{K}_1$ corresponds to $\phi(\partial^{-1}(\mathcal{K}_1)) = [0,1] \times [0,2] \times \cdots \times [0, n-1]$, i.e., the tuples such that $0 \leq \gamma_i < i$ for all i.

Proposition 8.9. *Let $\partial\sigma$ be in the dominant chamber and $\gamma = (\gamma_i) = \phi(\sigma)$ as above. Then, the length of $\psi(\sigma)$ is $\sum_i (n-i)\gamma_i$.*

Proof. We claim that $\sigma_1 + n > \sigma_j$ if and only if $\gamma_1 = \gamma_2 = \cdots = \gamma_{j-1} = 0$. Proof: Let U_i be defined as above for $1 \leq i \leq n-1$. Then, there are precisely $j-1$ elements of U_{j-1} which are greater than $\sigma_j - n$. Now, the following conditions are equivalent: (a) σ_1 is greater than $\sigma_j - n$; (b) $\sigma_1, \sigma_2, \ldots, \sigma_{j-1}$ are all greater than $\sigma_j - n$; (c) $\sigma_1, \sigma_2, \ldots, \sigma_{j-1}$ are the $j-1$ largest elements of U_{j-1}; (d) $\gamma_1 = \gamma_2 = \cdots = \gamma_{j-1} = 0$. This proves the claim.

We prove the proposition by induction on $\sum \gamma_i$. The result is obvious when all of the γ_i are zero, so we may assume this is not true. Let $\gamma_1 = \gamma_2 = \cdots = \gamma_{j-1} = 0$ and $\gamma_j > 0$. So $\sigma_j < \sigma_1 + n < \sigma_{j+1}$. Consider the element σ' defined by $(\sigma'_1, \sigma'_2, \ldots, \sigma'_n) = (\sigma_2 - 1, \sigma_3 - 1, \ldots, \sigma_j - 1, \sigma_1 + n - 1, \sigma_{j+1} - 1, \ldots, \sigma_n - 1)$. Then $\partial\sigma'$ is in the dominant chamber. It is not hard to verify that the vector $\gamma' = (\gamma'_1, \ldots, \gamma'_n) := \phi(\sigma')$ corresponding to σ' is $(0, 0, \ldots, 0, \gamma_j - 1, \gamma_{j+1}, \ldots, \gamma_{n-1})$. Let $t : \mathbb{Z} \to \mathbb{Z}$ denote the map $x \mapsto x+1$. Conjugation by t is a length preserving automorphism of $\widetilde{S}_n$ (where by length of σ, we mean $L(\psi(\sigma))$). Then, $t^{-1} \circ \sigma \circ t = \omega \circ \sigma'$, where ω is an $(n-j+1)$ cycle. It is not difficult to see that the RHS composition is length-additive, i.e., $L(\psi(\omega \circ \sigma')) = L(\psi(\omega)) + L(\psi(\sigma'))$ (e.g., one can use the standard formula for length, $L(\psi(\sigma)) = |\{(i,j) \mid i < j, 1 \leq i \leq n, \sigma(i) > \sigma(j)\}|$). Hence $L(\psi(\sigma)) = L(\psi(\sigma')) + (n-j)$. □

Hence, we have a bijection between the unit hypercube and $\{0\} \times \{0,1\} \times \cdots \times \{0,1,\ldots,n-2\}$ where the element corresponding to $(\gamma_1, \gamma_2, \ldots, \gamma_{n-1})$ has length $\sum (n-i)\gamma_i$. This gives a bijective proof that the rank generating function of the unit hypercube is $\prod_{i=1}^{n-1} \left(1 + t^{n-i} + t^{2(n-i)} + \cdots + t^{(i-1)(n-i)}\right)$.

We note that Eriksson and Eriksson also give combinatorial proofs of rank generating functions for other classical types; these proofs might be able to be similarly adapted.

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